Single-Particle Radiation Processes in a Plasma with Magnetic Fields*

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From the classical equation of motion, the radiation emitted by an electron in an external magnetic field undergoing Coulomb interactions is derived. It is shown that the several spectral components corresponding to magnetic and Coulomb force terms cannot be interpreted simply by saying that the total spectrum is composed of a cyclotron line superimposed on the continuum from the bremsstrahlung emission in the absence of the magnetic field.

1. INTRODUCTION

In previous papers,[1–3] we have shown that free electrons in the presence of ions and an external magnetic field emit a continuum with a sharp resonance at the cyclotron frequency. It is suggestive to identify the continuum with bremsstrahlung, admitting that its spectral features may be somewhat altered by the presence of the magnetic field, and to equate the resonance emission with the cyclotron line undergoing broadening effects by the electron-ion interactions.

This identification, however, is by no means self-evident. The quantum-mechanical treatment shows that the broad-band continuum and the resonance line both arise from the same solution of the wave equation, with no apparent distinction in the spectrum. In fact, the results suggest that there is no physical basis for the separation of the emission into two components. This is indeed the case.

It was stated in the introduction to Paper I that the whole problem of radiation due to Coulomb interactions in a magnetic field in principle could be obtained on the basis of classical theory, but that a quantum theory despite the cumbersome formalism is still easier to handle if the main interest is placed on obtaining accurate cross sections. Now, where these accurate cross sections are available for the whole spectrum, this argument is no longer valid, and a classical theory appears more adequate as well as simpler.

We derive in Sec. 2 the radiation spectrum from a classical equation of motion retaining appropriate damping and field terms as parameters. In this manner, we avoid the complications inherent in their explicit computations. We then proceed (Sec. 3) to calculate the radiated energy, coming back to the equation of motion and its representation of the interactions by force terms (Sec. 4). In Sec. 5, the initial value contribution is calculated. Finally, the results are discussed in Sec. 6.

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[1] R. Goldman and L. Oster, Phys. Rev. **129**, 1469 (1963).

[2] R. Goldman, Phys. Rev. **133**, A647 (1964).

[3] R. Goldman and L. Oster, preceding paper, Phys. Rev. **136**, A602 (1964).

2. SOLUTION OF THE EQUATION OF MOTION

The classical calculation of the radiation spectrum rests on computing the Fourier components of the acceleration to which the radiating particle is subjected. For this purpose, we have to solve the equation of motion which we write for a particle of charge q, mass m, and position vector $\mathbf{r}$, in the form

$$m\frac{d^2\mathbf{r}}{dt^2}=\frac{q}{c}\dot{\mathbf{r}}\times\mathbf{H}+\mathbf{F}-\nu\dot{\mathbf{r}}. \tag{1}$$

The phase terms on the right-hand side represent the interaction of the particle with a uniform external magnetic field $\mathbf{H}$, and the results of interactions with Coulomb fields which we split into a component $\mathbf{F}$ perpendicular to the instantaneous direction of motion and a term $\nu\dot{\mathbf{r}}$ parallel to the instantaneous velocity vector.

Making the z axis the direction of the magnetic field, we find for the x and y components of the acceleration

$$m\ddot{x}=(1/c)q\dot{y}H-\nu\dot{x}+F_x \tag{2}$$

and

$$m\ddot{y}=(-1/c)q\dot{x}H-\nu\dot{y}+F_y. \tag{3}$$

With

$$\rho=x+iy,\quad \Phi=F_x+iF_y, \tag{4}$$

we obtain

$$m\ddot{\rho}=(-i/c)qH\dot{\rho}-\nu\dot{\rho}+\Phi. \tag{5}$$

Equation (5) has the solution

$$\dot{\rho}=ae^{-\Omega T}+(1/m)e^{-\Omega T}\int_0^T e^{\Omega T}\Phi(t)dt, \tag{6}$$

where

$$\Omega=i\omega_c+\nu/m,\quad \omega_c=qH/mc, \tag{7}$$

and

$$a=v_x(0)+iv_y(0) \tag{8}$$

is a complex constant depending on the velocity of the particle at an initial time $t=0$. The acceleration $\ddot{\rho}$ is then obtained by differentiating Eq. (6) with respect to T.

3. RADIATED ENERGY

Returning to real variables with

$$\mathbf{r}_\perp = x\hat{\imath} + y\hat{\jmath}, \tag{9}$$

we have for the radiation emitted between time 0 and T

$$I(T) = \frac{2e^2}{3c^3}\int_0^T \ddot{\mathbf{r}}_\perp(t)\cdot\ddot{\mathbf{r}}_\perp(t)dt. \tag{10}$$

Next, we define the Fourier transform

$$\mathbf{r}(T) = (2\pi)^{-1/2}\int_{-\infty}^{+\infty} \mathbf{r}_\omega \exp(i\omega T)d\omega, \tag{11}$$

and

$$\mathbf{r}_\omega = (2\pi)^{-1/2}\int_{-\infty}^{+\infty} \mathbf{r}(t)\exp(-i\omega t)dt, \tag{12}$$

where we take $\mathbf{r}(t)=0$ for all $t<0$, since we are only concerned with displacements caused by the force $\mathbf{F}$ in the interval between initial time 0 and observing time T.

Differentiating Eq. (11) with respect to T and inserting the resulting expression into Eq. (10), we obtain

$$I(T) = \frac{2e^2}{3c^3}\int_0^T dt(2\pi)^{-1/2}\int_{-\infty}^{+\infty} d\omega(2\pi)^{-1/2}\int_{-\infty}^{+\infty} d\omega' \times\omega^2\exp(i\omega t)\omega'^2 \times\exp(i\omega' t)\{x_\omega x_{\omega'} + y_\omega y_{\omega'}\} + I_0(T), \tag{13}$$

where the contribution $I_0(T)$ is due to the first term in Eq. (6) which is independent of Φ and will be discussed separately in Sec. 5. Making use of the Fourier transform of the δ function,[4]

$$\frac{1}{2\pi}\int_0^\infty \exp[i(\omega+\omega')t]dt = \tfrac{1}{2}\delta(\omega+\omega'), \tag{14}$$

we can simplify Eq. (13), which transforms in the limit $T\to\infty$ to

$$I(T\to\infty) = \frac{2e^2}{3c^3} \times\tfrac{1}{2}\int_{-\infty}^{+\infty} \omega^4\{x_\omega x_{-\omega} + y_\omega y_{-\omega}\}d\omega + I_0(T). \tag{15}$$

We now express the Fourier components of x and y in terms of ω and F which is achieved with the aid of Eqs. (4) and (5). After some manipulations and introducing the complex conjugates

$$x_\omega^* = x_{-\omega},\quad y_\omega^* = y_{-\omega} \tag{16}$$

of x_ω and y_ω, we have

$$x_\omega = \tfrac{1}{2}F_{x,\omega}[\Omega_-^{-1}+\Omega_+^{-1}] + \tfrac{1}{2}iF_{y,\omega}[\Omega_-^{-1}-\Omega_+^{-1}] \tag{17}$$

and

$$y_\omega = -\tfrac{1}{2}iF_{x,\omega}[\Omega_-^{-1}-\Omega_+^{-1}] + \tfrac{1}{2}F_{y,\omega}[\Omega_-^{-1}+\Omega_+^{-1}]. \tag{18}$$

Here, $F_{x,\omega}$ and $F_{y,\omega}$ are the Fourier transforms of the x and y components of F, defined in the same manner as $\mathbf{r}_\omega$ in Eqs. (11) and (12),

$$\Omega_\pm = -m\omega^2 \pm m\omega\omega_c + i\nu\omega. \tag{19}$$

The radiated energy is now readily obtained in terms of ν and $\mathbf{F}$:

$$I(T\to\infty) = \frac{2e^2}{3c^3}\frac{1}{2}\int_0^\infty d\omega\{F,F^*\}\frac{\omega^2}{m^2}N + I_0(T), \tag{20}$$

where

$$N = [(\omega-\omega_c)^2+\nu^2/m^2]^{-1} + [(\omega+\omega_c)^2+\nu^2/m^2]^{-1}, \tag{20a}$$

and

$$\{F,F^*\} \equiv [F_{x,\omega}F_{x,\omega}^* + F_{y,\omega}F_{y,\omega}^*]. \tag{20b}$$

The starred quantities refer as before to the complex conjugates. Terms linear in $\mathbf{F}$, and terms of the form $F_{x,\omega}F_{y,\omega}^*$, etc., were neglected on the grounds that they cancel out in the average if the scattering ions that are responsible for the Coulomb fields represented by $\mathbf{F}$ are located at random. The randomness of ion location (and, for that matter, of all plasma correlations, dispersive effects, etc.) was one of the basic assumptions in the previous papers I–III. A similar argument incidentally was brought forth by Scheuer[5] in his treatment of bremsstrahlung.

Equation (20) describes the total amount of energy radiated by the electron in the presence of Coulomb scatterers and an external magnetic field. It thus corresponds to the expressions derived in I–III. For instance, if properly normalized to unit time, Eq. (20) becomes equivalent to Eq. (21) of paper III.

4. SEPARATION INTO MAGNETIC AND COULOMB CONTRIBUTIONS

Let us now turn back to Eq. (10) and treat the same problem once more, but making use of the force components introduced by Eq. (1):

$$(IT) = \frac{2e^2}{3c^3}\frac{1}{2}\int_0^T dt\left\{\frac{q}{mc}\dot{\mathbf{r}}\times\mathbf{H}\cdot\frac{q}{mc}\dot{\mathbf{r}}\times\mathbf{H} + \frac{1}{mm}\mathbf{F}(t)\cdot\frac{1}{m}\mathbf{F}(t) + 2\left(\omega_c - \frac{\nu}{m}\right)\dot{\mathbf{r}}\cdot\frac{1}{m}\mathbf{F}(t) + \frac{\nu^2}{m^2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}\right\} + I_0(T). \tag{21}$$

[4] Equation (14) in this form is, strictly speaking, incorrect, since the principal part contribution has been left out. This contribution is only then of importance if there are singularities in $(x_\omega x_{\omega'} + y_\omega y_{\omega'})$ for $\omega+\omega'\neq 0$. Examination of $x_\omega x_{\omega'}$ and $y_\omega y_{\omega'}$, however, shows that this is not the case as long as time intervals much longer than the reciprocal of the collision frequency ν from Eq. (1) are considered.

[5] P. A. G. Scheuer, Monthly Notices Roy. Astron. Soc. **120**, 231 (1960).

The first term can be reduced by the methods of the last section to the expression

$$I_1(T\to\infty)=\frac{2e^2}{3c^3}\frac{1}{2}\int_0^\infty d\omega\{F,F^*\}\frac{\omega_c^2}{m^2}N(\omega). \quad (22)$$

If $\nu/m\ll\omega_c$, only frequencies $\omega\approx\omega_c$ contribute significantly to the integral in (22) and the bracket containing terms such as

$$F_{x,\pm\omega}\approx F_{x,\pm\omega_c};\quad F_{y,\pm\omega}\approx F_{y,\pm\omega_c} \quad (23)$$

can be taken outside of the integral. Then,

$$I_1(T\to\infty)=\frac{2e^2}{3c^3}\frac{1}{2}\frac{\omega_c^2}{m^2}\{F,F^*\}\int N(\omega)d\omega, \quad (24)$$

which is of the form of a collision-broadened spectral line.

The second term reads simply

$$I_2(T\to\infty)=\frac{2e^2}{3c^3}\frac{1}{2}\cdot\frac{2}{m^2}\{F,F^*\}\int d\omega. \quad (25)$$

This term does not contain any resonance at all and thus describes a purely continuous spectrum. It reduces identically to the bremsstrahlung continuum in the absence of a magnetic field.

Obviously, these two contributions do not account for the total radiation emitted by the plasma. In fact, the third term

$$I_3(T\to\infty)=\frac{2e^2}{3c^3}\frac{1}{2}\int_0^\infty d\omega\frac{1}{m^2}\{F,F^*\}\times\left[\frac{2\omega_c(\omega-\omega_c)-2(\nu/m)^2}{(\omega-\omega_c)^2+(\nu/m)^2}-\frac{2\omega_c(\omega+\omega_c)+2(\nu/m)^2}{(\omega+\omega_c)^2+(\nu/m)^2}\right] \quad (26)$$

shows again a resonance for $\omega\approx\omega_c$. So does the fourth term,

$$I_4(T\to\infty)=\frac{2e^2}{3c^3}\frac{1}{2}\int_0^\infty d\omega\frac{1}{m^2}\{F,F^*\}\left(\frac{\nu}{m}\right)^2N(\omega), \quad (27)$$

which, if combined with (26), would merely reduce the insignificant terms $2(\nu/m)^2$ in the numerator by half.

Obviously, it must be true that

$$I(T)=I_0(T)+\sum_{i=1}^{4}I_i(T). \quad (28)$$

This identity is easily verified with the aid of Eq. (20).

5. CONTRIBUTION FROM THE INITIAL VELOCITY

Before we discuss these findings, we have to write out the contribution $I_0(T)$ that originates from the first term of Eq. (6), i.e., from

$$\ddot{\rho}_0=-(i\omega_c+\nu/m)\dot{\rho}_0,\quad \dot{\rho}_0=a\exp\{-(i\omega_c+\nu/m)T\}. \quad (29)$$

The radiated energy is found to be

$$I_0(T\to\infty)=\frac{2e^2}{3c^3}\frac{1}{2}(\omega_c^2+\nu^2/m^2)\times\int_{-\infty}^{+\infty}d\omega\{\dot{x}_\omega^0\dot{x}_{-\omega}^0+\dot{y}_\omega^0\dot{y}_{-\omega}^0\}, \quad (30)$$

where the Fourier components $\dot{x}_\omega^0$, etc., are defined with the aid of Eq. (4) replacing ρ by ρ_0.

Following the procedures of last section, we may write

$$I_0(T\to\infty)=\frac{2e^2}{3c^3}\frac{1}{2}(\omega_c^2+\nu^2/m^2)\times\{[v_x(0)]^2+[v_y(0)]^2\}\int_0^\infty N(\omega)d\omega, \quad (31)$$

where N is given again by Eq. (20a).

6. DISCUSSION

Let us first glance at I_0. The term does not depend on **F**, i.e., the part of Coulomb interactions that inflict changes on the instantaneous direction of velocity, nor directly on ν. Instead, it depends on the initial velocity. I_0 is in form and origin the "cyclotron line" derived on the basis of the simple-minded Lorentz theory.[6] For a term to term comparison, it is necessary to remember that our "intensities" are not normalized to unit time, that the collisions referred to in Ref. 6 are elastic [(hence, the proportionality to ω_c^2 instead of $(\omega_c^2+\nu^2/m^2)$], and that consequently our initial velocity v_0 remains unchanged and is identical with "the" velocity of the particle.

Our present assumption of random location of the scatterers is, in principle, built into the Lorentz-type theory as well, however, in the mathematical form of a randomness of phase changes *and* a randomness of times between collisions.[7]

The crucial point, however, is a comparison between I_0 and any one of the other resonance terms, in particular, I_1. Whereas I_0 stays finite even in the limit $T\to\infty$, since

$$\int_0^\infty N(\omega)d\omega=\pi m/\nu, \quad (32)$$

I_1 diverges for $T\to\infty$. This can be most easily seen by considering Eq. (20) and evoking Parseval's theorem.[8]

$$\int_0^\infty\{F,F^*\}d\omega\sim\int_0^T\mathbf{F}\cdot\mathbf{F}dt\sim\langle F^2\rangle_{\text{av}}T. \quad (33)$$

[6] L. Oster, Phys. Rev. **116**, 474 (1959), Eq. (14).
[7] Cf. The detailed discussion by L. Oster, Phys. Rev. **119**, 1444 (1960), Secs. 7 and 9.
[8] E. T. Whittaker and G. N. Watson, *A Course in Modern Analysis* (Cambridge University Press, New York, 1952), p. 182.

Hence, in the limit of a long observation time, the terms from Eq. (20) will dominate as expected.

These resonance terms represent the magnetic force alone (I_1), the Coulomb components parallel to the instantaneous velocity (I_4) and the hybrid force of both types of Coulomb interactions and the magnetic field (I_3). That they are proportional to FF^* is, at first, surprising until one realizes that the force terms are proportional to $\dot{\mathbf{r}}$, the instantaneous speed, which in turn depends on $\mathbf{F}$. Each of the terms adds a contribution to the resonance amplitude, and together they dominate at the resonance frequency over I_2. On the other hand, outside of that resonance, all four of the terms are essentially of equal weight. Clearly then, all four together represent the "bremsstrahlung" continuum outside of the line.

Letting the magnetic field go to 0 eliminates only I_1 (and, of course, part of I_0).[9] The relative magnitudes of the nonvanishing terms become

$$I_2/(I_3+I_4)\sim\omega/(\nu/m)\gg 1 \tag{34}$$

for most frequencies. Equation (34) can be interpreted by noting that the radiation produced by Coulomb interactions alone is originating predominantly in deflections from the electron path, and only to a minor degree in linear accelerations. This fact is well known in the theory of bremsstrahlung.[10] From this point of view, among all the Coulomb-induced terms, I_1 and I_2 will be dominant. This leads to another interesting observation: If we had neglected the damping term proper from the very beginning ($\nu\to 0$), we would have encountered a singularity at the resonance frequency, but no significant error further out. This type of approach is very common in kinetic theory, where Vlasov's equation is used instead of the complete Boltzmann equation.[11]

The preceding discussion makes it clear that the radiation emitted by electrons under the combined action of Coulomb scattering and external magnetic fields cannot be simply expressed as the sum of bremsstrahlung in the absence of a magnetic field (I_2) and cyclotron line emission (I_1).

The relative importance of the interference terms (I_3 and I_4) is illustrated in Fig. 1. We have plotted the ratio R of bremsstrahlung in the absence of a magnetic field (I_2) to the difference between total radiation and cyclotron line emission proper (I_1),

$$R=I_2/(I_2+I_3+I_4)\,. \tag{35}$$

[9] The corresponding limit would have been illegitimate in Refs. 6 and 7 due to the restrictions to the neighborhood of the resonance.

[10] L. Oster, Rev. Mod. Phys. **33**, 525 (1961).

[11] See, for instance, J. Dawson, and C. Oberman, Phys. Fluids **5**, 517 (1962).

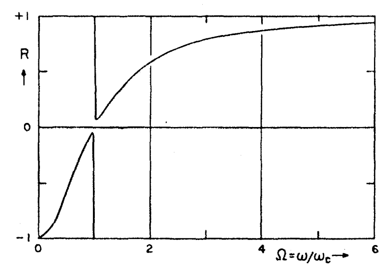

FIG. 1. Ratio of field-free bremsstrahlung intensity to that of total radiation minus cyclotron line contribution, as a function of frequency in units of the cyclotron frequency.

Neglecting for simplicity the damping terms ($\nu\to 0$, $I_4\to 0$), and introducing the parameter

$$\Omega=\omega/\omega_c\,, \tag{36}$$

i.e., the frequency in units of the cyclotron frequency, we have

$$R=2\left\{\frac{\Omega+1}{\Omega-1}+\frac{\Omega-1}{\Omega+1}\right\}^{-1}. \tag{37}$$

Equation (37) holds except very close to the resonance $\omega\to\omega_c$, $\Omega\to 1$ where it predicts a zero that is physically unreasonable. Inspection of the complete solution with finite ν reveals that

$$R\approx(\nu^2/m^2\omega_c^2)[\omega-\omega_c]^{-1}\,,\quad \omega\approx\omega_c\,, \tag{38}$$

i.e., R goes to infinity at the resonance. The combination of Eqs. (37) and (38) is plotted in Fig. 1.

The ratio R shows a rather interesting and to some extent unexpected behavior; for frequencies below the resonance, the ratio is negative, that is, the cyclotron term proper I_1 is larger than the total radiation emitted. It was already mentioned above that in the low-frequency limit, the interference terms have the greatest influence on the result. For frequencies above the resonance, the ratio approaches +1. Physically, this behavior mirrors the fact that at high frequencies, the emission is only insignificantly affected by the presence of a magnetic field, and that the total radiation is thus practically equal to the bremsstrahlung term. At the resonance, finally, R goes to infinity, i.e., the total radiation equals the contribution from the resonance term proper, the "bremsstrahlung term" I_2 is canceled by the two interference terms.

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